ENME 408 Spring 2026
Project 1

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Honor code:
"I have neither received nor given aid to anyone outside my team during this project."

1. **Derive the multicopter equations of motion (EOM).**

   (a) **Frames and angles.** Define the inertial frame F_A (use NED) and body frame F_D, and the intermediate frames. Write the orientation matrices that relate the various frames.

   Answer: Using NED to define the inertial frame, F_A, we can write the Euler rotations to the body-fixed frame, F_D, as follows:

$$F_A \xrightarrow[3]{\psi} F_B \xrightarrow[2]{\theta} F_C \xrightarrow[1]{\phi} F_D.$$

   Where F_B and F_C are intermediate frames. The Euler orientation matricies are:

$$\mathcal{O}_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$\mathcal{O}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathcal{O}_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (b) **Position, velocity, and acceleration.** Let c denote the center of mass of the multicopter and let w be a fixed point fixed on Earth. Let $\vec{r}_{c/w} \triangleq p_1\hat{\imath}_A + p_2\hat{\jmath}_A + p_3\hat{k}_A$. Write the expressions for the physical velocity vector $\vec{v}_{c/w/A}$ and the physical acceleration vector $\vec{a}_{c/w/A}$ in terms of p_1, p_2, p_3, their derivatives, and the inertial frame F_A axes.

   Answer: To find the velocity of the center of mass of the multicopter, c, relative to w and with respect to F_A, we can take the frame derivative of $\vec{r}_{c/w}$.

$$\overset{A\bullet}{\vec{r}}_{c/w} \triangleq \vec{v}_{y/w/A} = \overbrace{p_1\hat{\imath}_A}^{A\bullet} + \overbrace{p_2\hat{\jmath}_A}^{A\bullet} + \overbrace{p_3\hat{k}_A}^{A\bullet}$$

$$= \dot{p}_1\hat{\imath}_A + p_1\overset{A\bullet}{\hat{\imath}}_A + \dot{p}_2\hat{\jmath}_A + p_2\overset{A\bullet}{\hat{\jmath}}_A + \dot{p}_3\hat{k}_A + p_3\overset{A\bullet}{\hat{k}}_A$$

   And since $\overset{A\bullet}{\hat{\imath}} = \overset{A\bullet}{\hat{\jmath}} = \overset{A\bullet}{\hat{k}} = 0$,

$$\boxed{\vec{v}_{y/w/A} = \dot{p}_1\hat{\imath}_A + \dot{p}_2\hat{\jmath}_A + \dot{p}_3\hat{k}_A}$$

   Then, to find the physical acceleration vector,

$$\vec{a}_{c/w/A} = \overset{A\cdot}{\vec{v}}_{y/w/A}$$

So,

$$\boxed{\vec{a}_{c/w/A} = \ddot{p}_1\hat{\imath}_A + \ddot{p}_2\hat{\jmath}_A + \ddot{p}_3\hat{k}_A}$$

(c) **Equations of motion.** Let the gravity vector be given by $\vec{g} = g\hat{k}_A$ and consider a multicopter with mass m and inertia matrix $\vec{\boldsymbol{J}}_{\mathcal{B}/c}\Big|_D = J = \text{diag}(J_1, J_2, J_3)$ that can exert a thrust $\vec{T} = -T\hat{k}_D$ applied at c, where $T \geq 0$, and a torque $\vec{\tau} = \tau_1\hat{\imath}_D + \tau_2\hat{\jmath}_D + \tau_3\hat{k}_D$, where $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$. Using Newton's 2nd law, Poisson's equation, and Euler's equation, write the equations of motion of the multicopter. In particular, obtain expressions for $[\ddot{p}_1\ \ddot{p}_2\ \ddot{p}_3]^T$, $[\dot{\psi}\ \dot{\theta}\ \dot{\phi}]^T$, $[\dot{\omega}_1\ \dot{\omega}_2\ \dot{\omega}_3]^T$, where ψ, θ, ϕ are the 3-2-1 Euler angles and $\omega_1, \omega_2, \omega_3$ are the body fixed angular velocities.

Answer: First, we can find an equation linking the forces acting on the multicopter,

$$\sum \vec{f} = \vec{f}_{\mathcal{B}} = \vec{T} + m\vec{g} = -T\hat{k}_D + mg\hat{k}_A$$

And do the same for the total moment,

$$\vec{M}_{\mathcal{B}/c} = \vec{\tau}_{\mathcal{B}} = \tau_1\hat{\imath}_D + \tau_2\hat{\jmath}_D + \tau_3\hat{k}_D$$

Using Newton's 2nd law:

$$-T\hat{k}_D + mg\hat{k}_A = m\vec{a}_{c/w/A}$$

$$-T\hat{k}_D + mg\hat{k}_A = m\left(\ddot{p}_1\hat{\imath}_A + \ddot{p}_2\hat{\jmath}_A + \ddot{p}_3\hat{k}_A\right)$$

Resolving the above in the inertial frame, F_A, also knowing $\hat{k}_D\Big|_A = \mathcal{O}_{A/D}\hat{k}_D\Big|_D$ gives us:

$$-T\mathcal{O}_{A/D}\hat{k}_D\Big|_D + mg\hat{k}_A\Big|_A = m\ddot{p}_1\hat{\imath}_A\Big|_A + m\ddot{p}_2\hat{\jmath}_A\Big|_A + m\ddot{p}_3\hat{k}_A\Big|_A$$

$$-T\mathcal{O}_{A/D}\boldsymbol{e}_3 + mg\boldsymbol{e}_3 = m\ddot{p}_1\boldsymbol{e}_1 + m\ddot{p}_2\boldsymbol{e}_2 + m\ddot{p}_3\boldsymbol{e}_3$$

Then dividing by m gives us:

$$\boxed{\vec{a}_{c/w/A} = \begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \ddot{p}_3 \end{bmatrix} = \mathcal{O}_{A/D} \begin{bmatrix} 0 \\ 0 \\ -T/m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}}$$

The orientation matrix could also be rewritten slightly:

$$\mathcal{O}_{A/D} = \mathcal{O}_{D/A}^T = \Big(\mathcal{O}_1(\phi)\mathcal{O}_2(\theta)\mathcal{O}_3(\psi)\Big)^T$$

Then using Euler's equation about the center of mass,

$$\vec{\boldsymbol{J}}_{\mathcal{B}/c} \overset{D\cdot}{\vec{\omega}}_{D/A} + \vec{\omega}_{D/A} \times \vec{\boldsymbol{J}}_{\mathcal{B}/c} \vec{\omega}_{D/A} = \vec{M}_{\mathcal{B}/c}$$

And using kinematics, $\vec{\omega}_{D/A} = \omega_1 \hat{\imath}_D + \omega_2 \hat{\jmath}_D + \omega_3 \hat{k}_D$, $\overset{D\cdot}{\vec{\omega}}_{D/A} = \dot{\omega}_1 \hat{\imath}_D + \dot{\omega}_2 \hat{\jmath}_D + \dot{\omega}_3 \hat{k}_D$, and resolving each term in F_D, we can find:

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

Next, the above equation can be rewritten to have $\overset{D\cdot}{\vec{\omega}}_{D/A}$ on one side.

$$\overset{D\cdot}{\vec{\omega}}_{D/A} = \boxed{\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{bmatrix} \left[\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} - \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left(\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) \right]}$$

Now we have the equations of motion for $[\ddot{p}_1\ \ddot{p}_2\ \ddot{p}_3]^T$ and $[\dot{\omega}_1\ \dot{\omega}_2\ \dot{\omega}_3]^T$, next, to get the expressions for $[\dot{\psi}\ \dot{\theta}\ \dot{\phi}]^T$, we can use the Euler-Angle derivations from the textbook (3.7.12), following the equation $\dot{q} = S(\phi, \theta)^{-1} \omega_{D/A|D}$.

$$\boxed{\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & (\sin\phi)\tan\theta & (\cos\phi)\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & (\sin\phi)\sec\theta & (\cos\phi)\sec\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}$$

(d) **State-space form.** Write the equations of motion in the state-space form $\dot{x} = f(x, u)$, where

$$x \triangleq [p_1 \quad p_2 \quad p_3 \quad \dot{p}_1 \quad \dot{p}_2 \quad \dot{p}_3 \quad \psi \quad \theta \quad \phi \quad \dot{\omega}_1 \quad \dot{\omega}_2 \quad \dot{\omega}_3]^T \ , \ u \triangleq [T \quad \tau_1 \quad \tau_2 \quad \tau_3]^T.$$

Answer: Using the equations found in part (c), we can write the equations as follows:

$$
\underbrace{\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \ddot{p}_1 \\ \ddot{p}_2 \\ \ddot{p}_3 \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \Big(\mathcal{O}_1(\phi)\mathcal{O}_2(\theta)\mathcal{O}_3(\psi)\Big)^T \begin{bmatrix} 0 \\ 0 \\ -T/m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \\ \begin{bmatrix} 1 & (\sin\phi)\tan\theta & (\cos\phi)\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & (\sin\phi)\sec\theta & (\cos\phi)\sec\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \\ \begin{bmatrix} 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{bmatrix} \left[\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} - \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left(\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) \right] \end{bmatrix}}_{f(x,u)}
$$

2. **Simulate the EOM in Simulink.**

   (a) **Model structure.** Implement the equations of motion using vectors and matrices to compute $\dot{x}$. Use the integrator blocks to obtain the state x for $t \in [0, 10]$s in the case where $x(0) = 0$ and $u(t) = [mg + \sin(\pi t) \quad 0 \quad 0 \quad 0]^T$. Plot the position, velocity, and the Euler angles versus time.

   Hint: Use a fixed-step solver (e.g., RK4) with a small step size (e.g., $\Delta t = 10^{-4}$ s).

Answer:

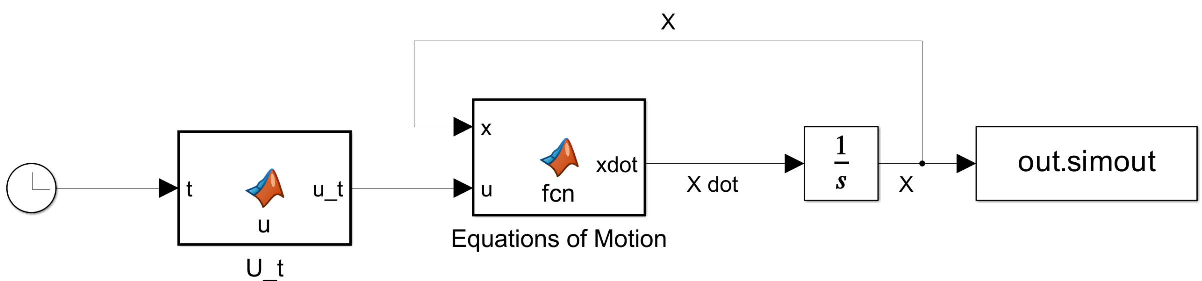

Figure 1: EOM Simulink Model

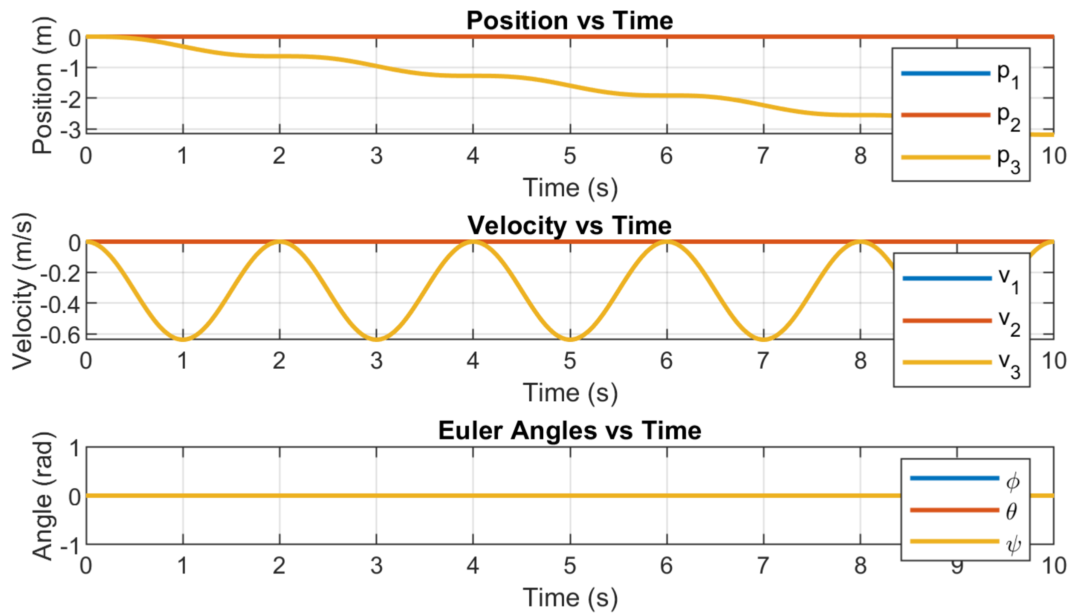

Figure 2: EOM Simulation Results

Figure 2 demonstrates that the simulation results are consistent with the derivations obtained in Part 1.

Beginning with the *Position Vs Time* plot, which corresponds to the first three states of the system, we observe that p_1 and p_2 remain constant while p_3 decreases over time. This confirms that the thrust is less than the gravitational force acting on the system. From the translational dynamics

$$\ddot{p}_3 = g - \frac{T}{m}$$

Which implies that when $\frac{T}{m} < g$, the vertical acceleration is negative and the vehicle moves downward, as observed in the plot.

The second plot, *Velocity vs Time*, matches the translational dynamics. Since Euler angles satisfy

$$\phi = \theta = \psi = 0$$

the rotation matrix simplifies to the identity matrix. Consequently, the translational acceleration only affects the vertical velocity component v_3, while v_1 and v_2 remain zero.

Finally, the angular velocity plot shows that

$$\omega_1 = \omega_2 = \omega_3 = 0$$

which indicates that there is no rotational motion. Because the rotation reduces to the identity matrix, the motion occurs only in the vertical direction.

3. **Open-loop control.**

(a) Let $m = 1$ and $J = \text{diag}(0.01, 0.01, 0.02)$. Consider the desired trajectory

$$p_1(t) = \cos(\Omega t), \quad p_2(t) = sin(\Omega t), \quad p_3(t) = 0,$$

with $\Omega = 0.1 \times 2\pi$ rad/s. Using the translational equations of motion, compute the open-loop force vector $\vec{f} = f_1 \hat{\imath}_A + f_2 \hat{\jmath}_A + f_3 \hat{k}_A$ required for the quadrotor to track this circular trajectory. Plot $p_1(t)$, $p_2(t)$, and $p_3(t)$ over time.

i. Use `plot3(x,y,z)` to show the position path in NED coordinates.
Answer:

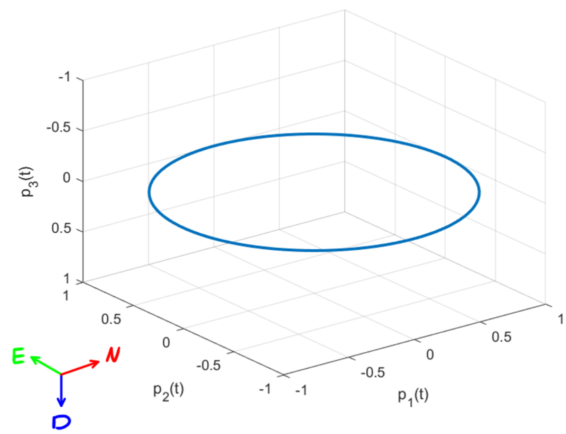

Figure 3: MATLAB `plot3(`p_1, p_2, p_3`)` in NED coordinates

ii. **Vector overlay.** Use `quiver3` to overlay velocity vectors along the trajectory.

A. Downsample the vectors (e.g., every 10–50 samples) so the plot stays readable.
Answer:

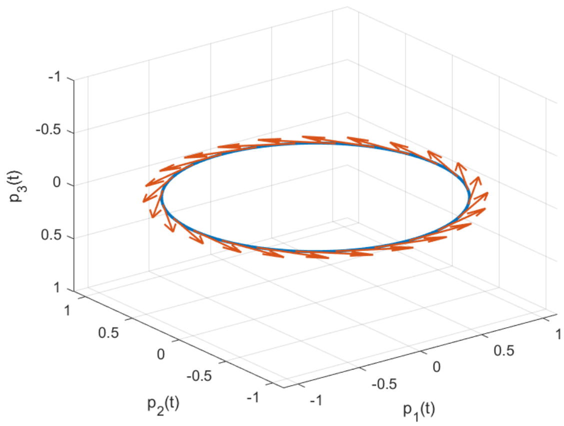

Figure 4: MATLAB `quiver3` plot

B. Apply a scaling factor so arrows are visible but not overwhelming. Next, plot f_1, f_2, and f_3 versus time in a single plot.
Hint: Recall that $\vec{f} = m\vec{a}_{c/w/A}$.
Answer:

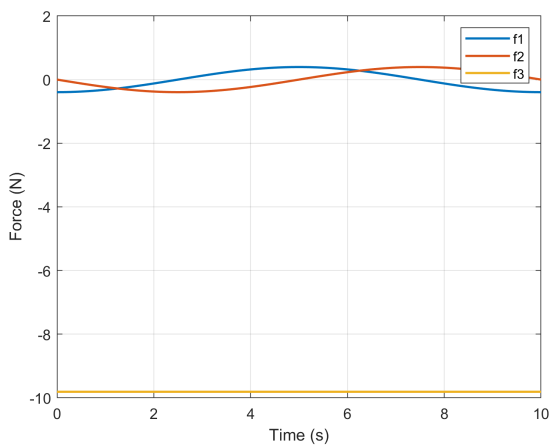

Figure 5: MATLAB forces plot

(b) Assume $\psi(t) = 0$. Compute the roll and pitch angles $\phi(t)$ and $\theta(t)$ required to realize the desired force vector $f(t)$. Plot $\phi(t)$ and $\theta(t)$ as functions of time. **Hint:** First, obtain an expression for T. Then, use Newton's Second Law to obtain $\phi(t)$ and $\theta(t)$. **Answer:**

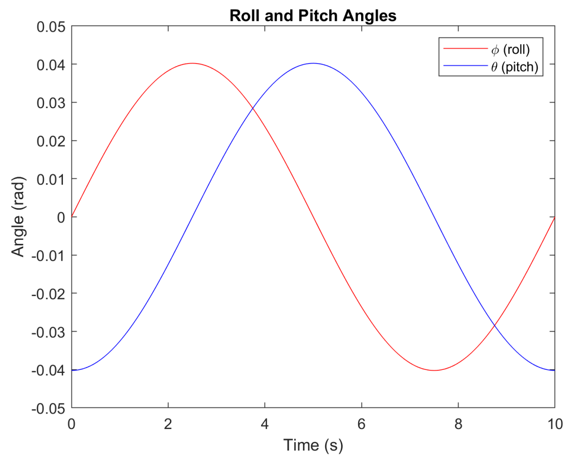

Figure 6: MATLAB Roll and Pitch Angles Plot

(c) Using Poisson's equation, compute the body-fixed angular velocity vector $\omega_{D/A|D}(t)$ corresponding to the Euler angle trajectories. Plot the components of $\omega_{D/A|D}(t)$ over time.

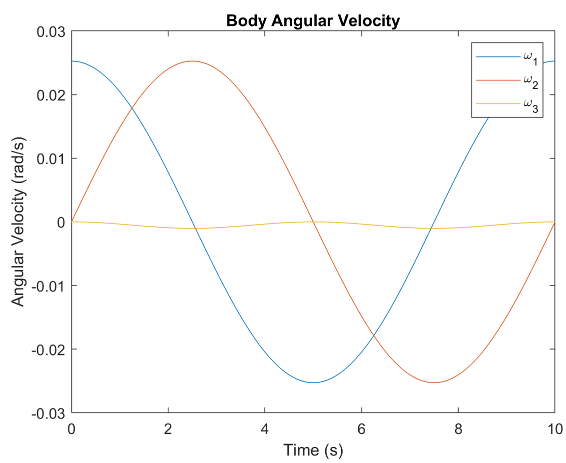

Figure 7: MATLAB Body Angular Velocity Plot

(d) Using Euler's rotational equations of motion, compute the control torque $\tau(t)$ required to generate the angular velocity $\omega_{D/A|D}(t)$. Plot the components of $\tau(t)$ over time.

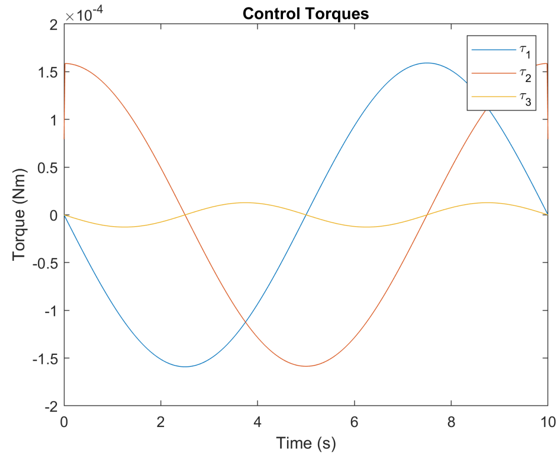

Figure 8: MATLAB Control Torques Plot

(e) Simulate the full quadrotor dynamics under the computed open-loop thrust T and torques τ_1, τ_2, τ_3 for $t \in [0, 10]$ s and confirm whether the resulting trajectory tracks the desired circular reference. Use appropriate initial conditions.

Answer:

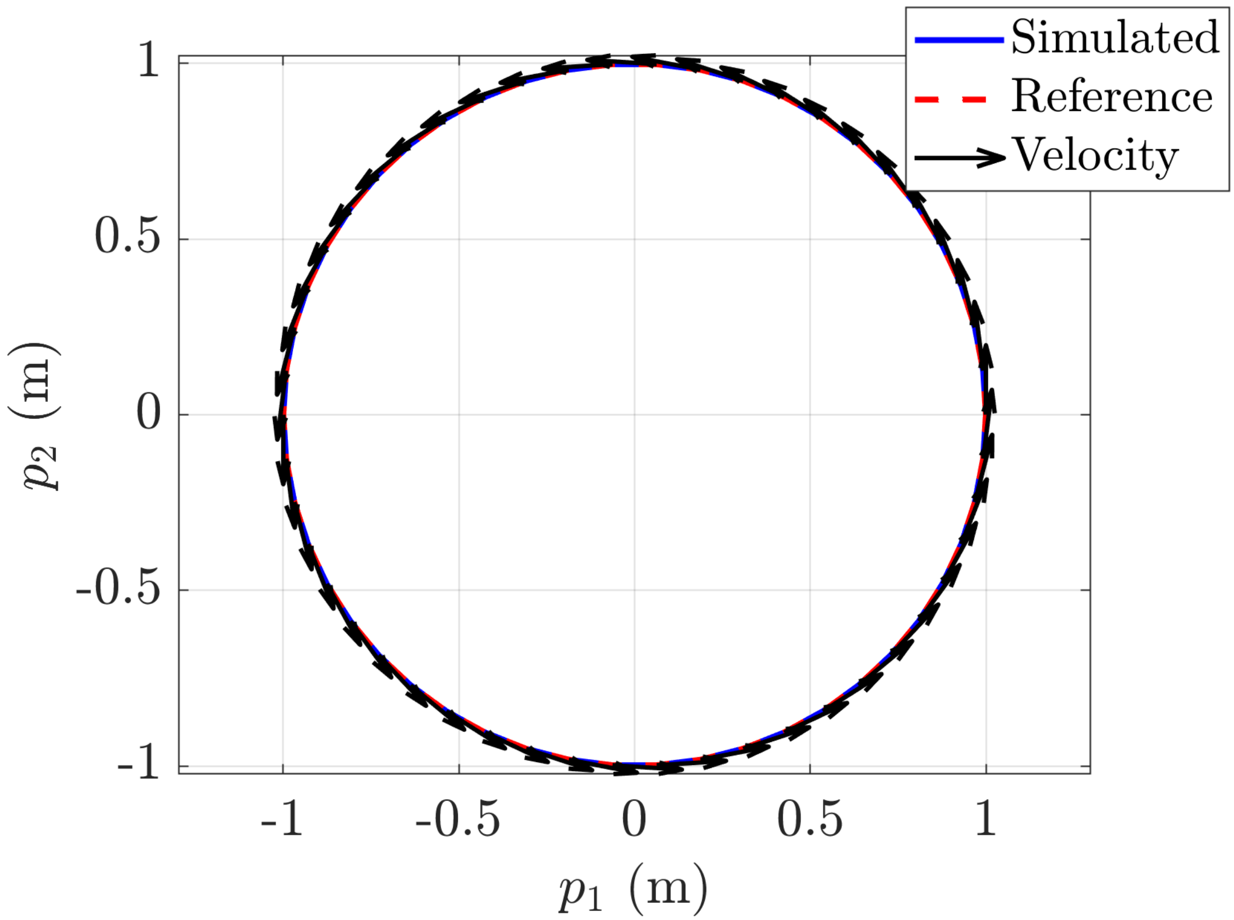

Figure 9: Trajectory Tracking of Quadrotor under Open-Loop conditions

(f) **Effect of perturbations.**

i. Add a small random noise to the initial position, that is,
p1(0) = 1 + w1, p2(0) = w2, p3(0) = w3, (3) where wi are random numbers generated in Matlab using randn. With the perturbed initial position, simulate the full quadrotor dynamics using the previously computed open-loop forces and torques for t [0, 10] s. Comment on the effect of initial position perturbations on the position deviations from the desired circular trajectory.
Answer:

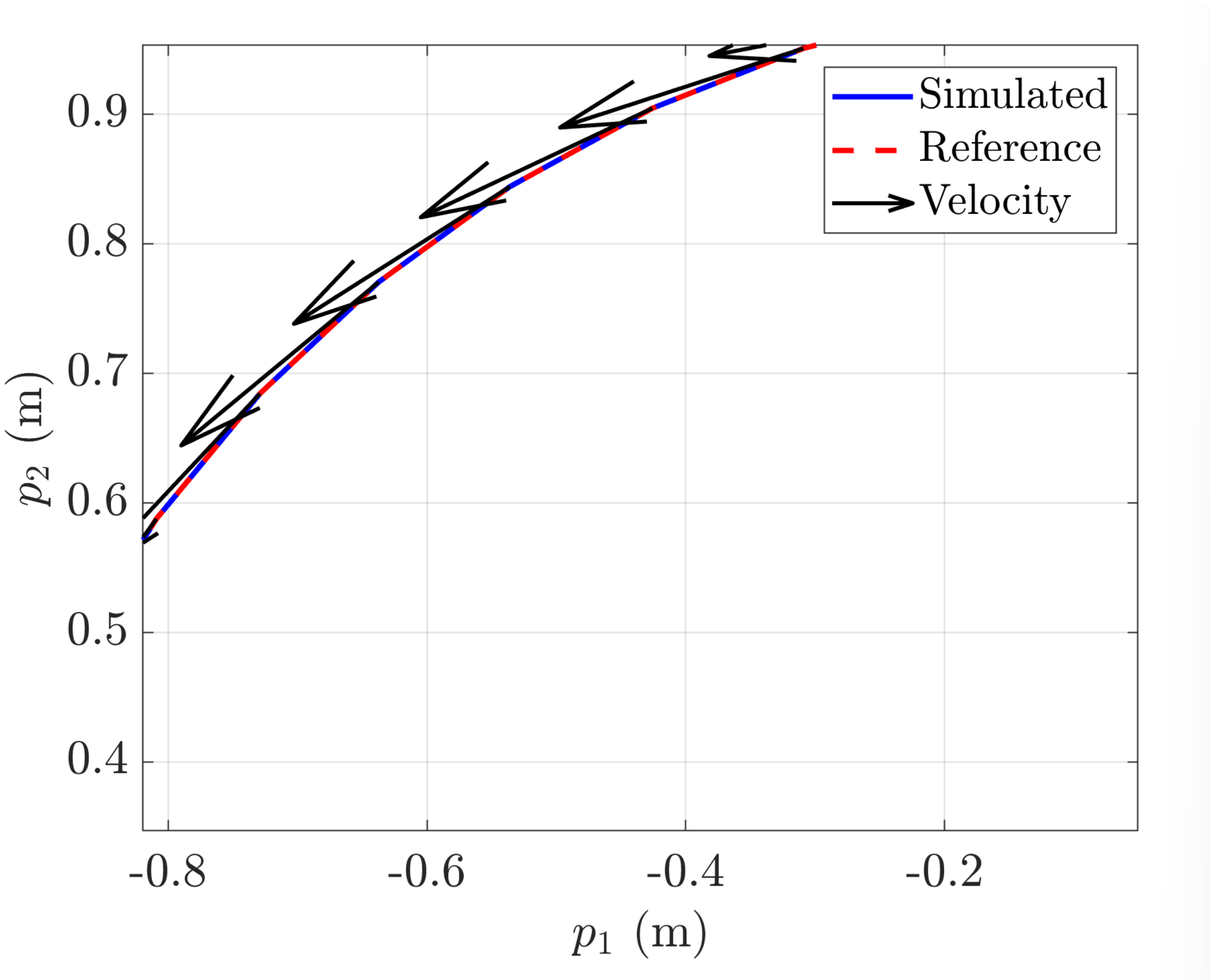

Figure 10: Nominal Case of Quadrotor under Open-Loop conditions

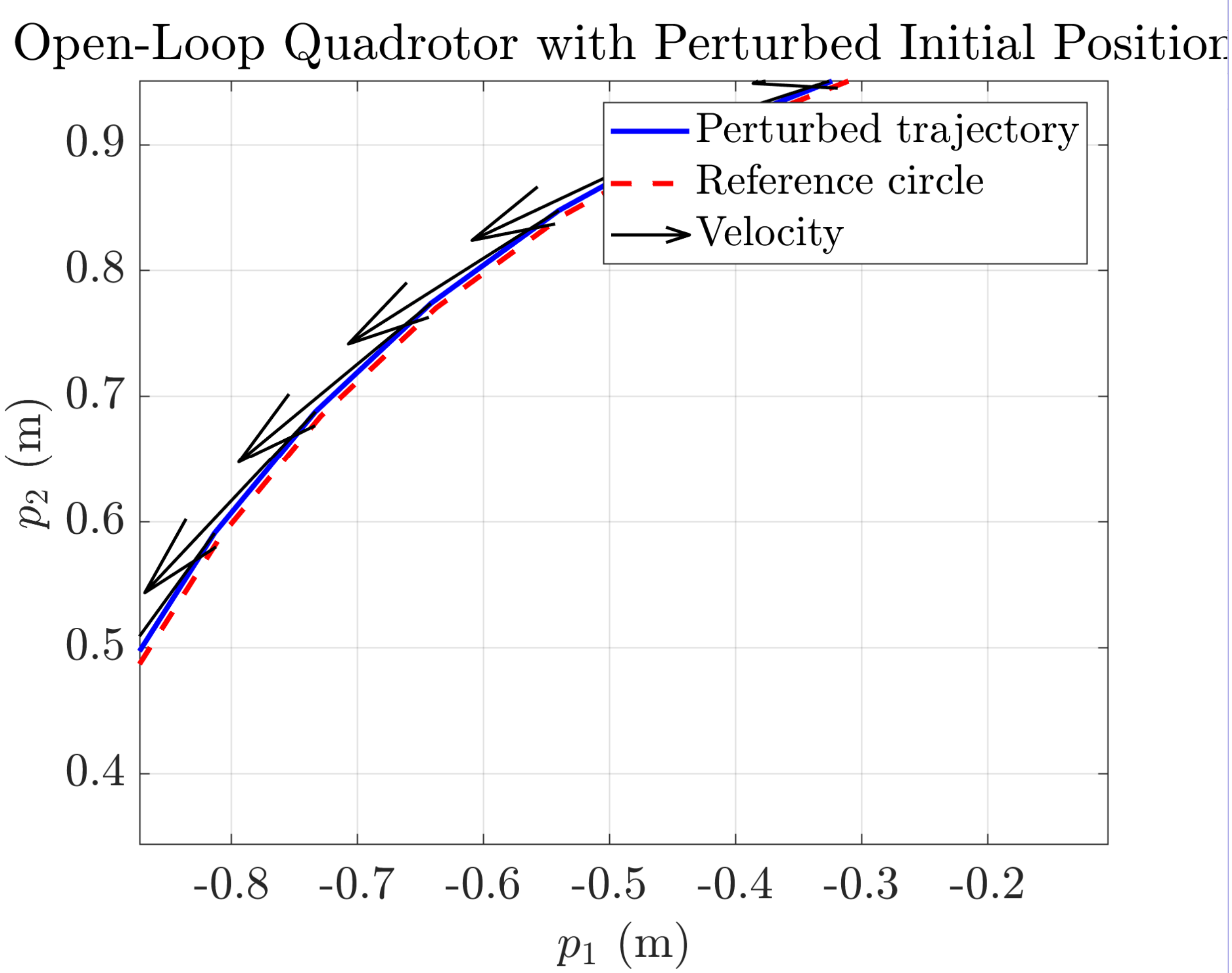

Figure 11: Positionally Perturbed Case of Quadrotor under Open-Loop conditions

The quadrotor starts slightly off the desired circular trajectory. Because the system is open-loop, there is no feedback to correct the deviation. Therefore, the quadrotor will continue following a path shifted to the reference circle. Since the perturbation is small, the derivation is barely noticeable.

ii. Add a small random noise to the initial velocities, that is,
p 1(0) = w1, p 2(0) = + w2, p 3(0) = w3, (4) where wi are random numbers generated in Matlab using randn. With the perturbed initial velocity, simulate the full quadrotor dynamics using the previously computed open-loop forces and torques for t [0, 10] s. Comment on the effect of initial velocity perturbations on the position deviations from the desired circular trajectory.

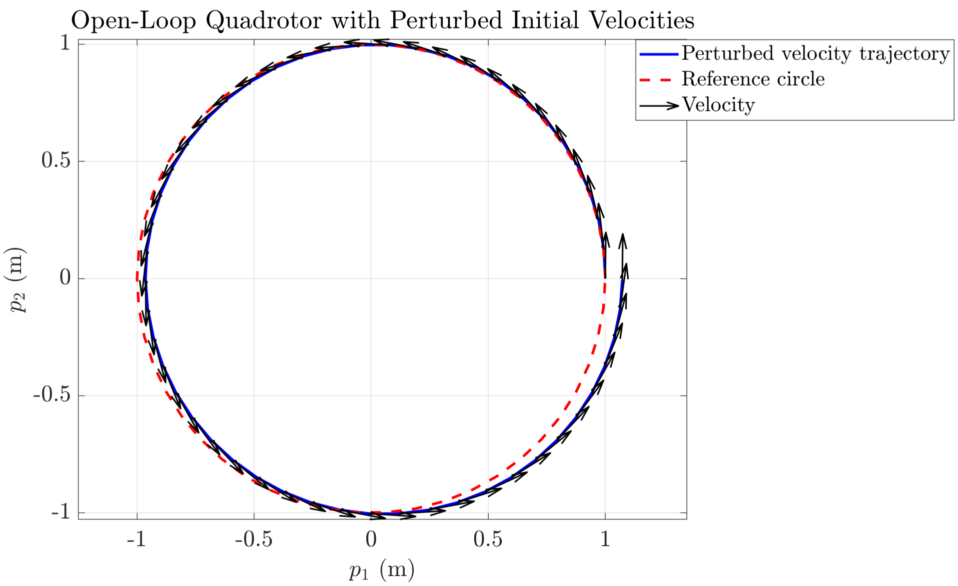

Figure 12: Initial Velocity Perturbations of Quadrotor under Open-Loop conditions

Answer: When you perturb the initial velocities, larger deviations are produced over time, like in this case where the trajectory of the quadcopter overshot the reference circle. While sigma was small, large deviations were observed, compared to the nominal case and positionally perturbed case.

Credit Author Statement

Christopher Appiah: Simulations for Question 2, Question 3 Part (e), and Question 4. MATLAB Plotting and Computation for Question 3 Parts (b)-(d)
Ethan Hardesty: Question 1 All Derivations and Processes, MATLAB Plotting and Computation for Question 3